Optimization In Structural Analysis And Design

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ABSTRACT

Two main tasks of a structural engineer, as for many other branches of engineering, are analysis and design. Among these two, the latter needs more knowledge, skill and experience. It even comprises completely the first one, that is, a designer must already have the capacity of analysis.

There are two technical tools to be used in these two tasks. One of them is finding solution(s) to equilibrium equations of the form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ and $\mathbf{b}$ are known, $\mathbf{x}$ is unkown. For nonlinear problems this equation becomes nonlinear without changing the main character of the problem: find $\mathbf{x}$ satisfying the nonlinear equations put forth. The second tool is optimization, i.e. to find $\mathbf{x}$ making $\Phi(\mathbf{x})$ optimum, either a maximum or minimum.

If one is asked to allocate tools to tasks, the answer generally will be such that two pairs are to be formed: (analysis, equilibrium conditions by root finding), (design, optimization). In fact, this allocation can best be shown by forming a two by two matrix. In this matrix, the two locations corresponding to the pairs mentioned above will be the ones that have the highest usage. The location corresponding to (design, equilibrium) will not be empty because most design problems are solved after some successive equilibrium applications. The least used location will be (analysis, optimization).

In fact, there seems to be a mistake in this emptiness of the location corresponding the the analysis of structures by using a formulation which based on optimization, because we all know that a structure, loaded in whatever manner, will take the configuration having the least total potential energy. Thus, to analyze a structure under some loading, it suffices to find its shape with the least potential. Once this shape is determined, all the displacements, deformations, strains, stresses, reactions etc. can be determined to complete the analysis.

This idea is now gaining importance and enabling engineers to solve problems which were very difficult to solve by the accustomed methods. This means that optimization is enlarging its range of applicability from design part to analysis part. If one considers in this context the advantage of new techniques of optimization, like metaheuristic methods, one will realize that optimization will increase its importance in structural engineer’s life both in analysis and design.

INTRODUCTION

Optimization is in general characterized by three elements:

- A set of n independent variables $x_1, x_2, ..., x_n$, forming the vector $\mathbf{x}$, or the **chromosome**,
- A set of functions to be optimized, $f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_m(\mathbf{x})$, which are the **objective function**(s),
- A set of constraints to be satisfied, $g_1(\mathbf{x}) \leq 0, g_2(\mathbf{x}) \leq 0, ..., g_s(\mathbf{x}) \leq 0$.

It is to be noted that some or all of the constraints may be equalities.

Then an optimization problem is stated as the determination of **x**, to optimize the functions **f** in such a way that the constraints **g** =≤ **0** will be satisfied.

The followings are basic information about optimization problems.

Maximization, minimization. Optimization is either maximization or minimization. In the following, unless otherwise stated, we will consider minimization problems, that is we will consider problems as

$$\min \mathbf{f(x)}, \text{ find } \mathbf{x}.$$

This does not create any loss of generality since maximization problems and minimization problems can be interchanged by a negative sign in front of the objective function(s).

Single variable – multi variable optimization. In some problems, the set **x** may consist of one single variable. This will form a single variable optimization problem as compared to multi variable optimization problems.

Discrete optimization – Continuous optimization. The unknown set **x**, may be formed of discrete variables or continuous variables depending on the formulation of the problem. For instance for the same problem of truss weight optimization, the members may be chosen from a list already at hand, or their characteristics such as the cross sectional area may be the unknowns. In the first case the answer will be something like "3rd in the list", while for the second case the answer would be something like "102.34 mm^2". In the first case the answer will be chosen among a finite number of options, while for the second case the choice will be done among a continuous range providing infinitely many alternatives. Accordingly, the flow chart of the operations will be quite different.

Single objective – multi objective optimization. It is very rare in real life to encounter a problem with a single objective. Very often, problems pose themselves with two or more objectives, classical example being minimization of the cost and minimization of accidents per unit product. As in this example, the objectives are usually contradictionary: at the limit, one cannot reduce the cost and the number of accidents at the same time. In such cases, there is room for trade-off between objectives and this ends with a definition called **Pareto optimality**. The simplest way to deal with a multi objective problem is to form a combined objective function

$$F = w_1f_1 + w_2f_2 + \ldots + w_mf_m$$

to be minimized where w_1, w_2, … , w_m are the weighing parameters showing the importance attributed to different objectives. Depending on the choices of w_i's, one will find in general infinite number of probable solutions, forming the set of Pareto solutions.

Local and global optima. In an optimization problem, another rare phenomen is the uniqueness of the optimum point. In general, there are many points which are better in their neighborhoods, so they are called local optimum points. The best among them is the global optimum. All optimization methods have the deficiency of not being able to differentiate between a local optimum point and the global optimum one, though some of them have measures to move more or less to the global one.

Unconstrained and constrained optimization. Constrained problems, introduced at the beginning, are much more complex than unconstrained problems where there are no relations of type **G** (**x**) ≤ **0** to be satisfied. The vectors **x** that satisfy the constraints form the **set of feasible vectors**. In some problems it may be easy to find these feasible vectors thus one can search for the optimum one among them. In most of the cases it may not be easy to determine them, so the optimality and feasibility can be searched together by minimizing the **fitness function**

$$U = F + P$$

where P is the **penalty function** calculated by measuring the degrre of violance of constraints. In such a case, a solution will be acceptable if the final solution at hand has zero penalty.

OPTIMIZATION TECHNIQUES

Classical optimization techniques.

These are methods that were in use before the emergence of the so-called metaheuristic methods. These methods start from calculus methods, beginning from equating the derivative to zero, and go to methods called nonlinear programming, dynamic programming, integer programming, etc (Note that the word "programming" here is used almost in the same sense of "optimization"). These methods are rarely direct, generally iterative. These methods are generally not exhaustive, i.e. they are applicable only to those problems for which they are formulated. Some of them requires the functions to be differentiable up to a certain order. Only a few of them can tackle constraints.

Metaheuristic optimization techniques. These methods owe their emergence mainly to two developments: Advances in capacity of computers, and, Advances in the look of mathematicians and scientists to the physical world and nature

In fact, it may be said that these two factors have triggered each other mutually, and gave way to terms like "intelligent computing" and "soft computing". The first factor is related especially to memory capacity and speed of computers, but also to the advances in computer languages towards object oriented programming. The second factor is related with intelligent look to living and not living nature, and imitating the optimization processes existing thereof. This look gave way first to methods called Genetic Algorithms, and Simulated Annealing, and then to Ant Colony Optimization, Swarm Particle Optimization, Big Bang and Big Crunch Method, Harmony Search, Tabu Search and many others obtained by hybrid applications of these methods. It is to be noted that thses methods involve so many computations that without the named advances in computer sciences, they would not be practically applicable.

Obvious advantages of these methods over classical ones can be named as:

- Their generality,
- Ease of tackling problems with functions not differentaiable, even not continuous or smooth,
- Ease of tackling problems with functions discrete or continuous,
- Ease of tackling problems with constraints
- Facilities for finding global optima, though without guarantee

- Nonlinearity of functions to be optimized, or the constraints, or the penalty functions do not pose insurmountable theoretical problems, they just add some function evaluation difficulties.

The only disadvantage of these methods is the extra CPU time often needed. But this is acceptable seen the fact that problems can be solved by these methods which were otherwise unsolvable.

STRUCTURAL OPTIMIZATION

In Haftka and Gürdal [1992] a general overview of classical methods of structural optimization is given. Linear programming and sequential linear programming, with changing limits at each iteration, takes an important place among them. Finite element programs integrated with such techniques are shown to be effective in solving quite complicated problems.

Frangopol and Cheng [1997] have collected quite a number of articles about structural optimization. Among them, one can see the introduction of Genetic Algorithms besides classical methods for solving structural optimization problems. Large space structures are dealt with by Kamat [1985] and a general methodology for shape optimization is given by Bugeda and Oliver [1992].

Metaheuristic methods became very popular in the last decades. Fundamentals of energy considerations in structures can be found in Oden [1967]. General notes about metaheuristic methods can be found in Hatay and Toklu [2002] and in Toklu [2004 a, b, c, d]

ENGINEERING PROBLEMS AND SOLUTION TECHNIQUES

Engineering problems are generally of two types: Analysis and Design. In analysis problems an engineer is asked to analyse a given system under the effect of generalized effects, physical loads, temperature differences, electrical charges, chemical attacks, etc. Here the system and the generalized loads are given, the behavior of system is questioned as to reactions, internal forces, displacements, stresses, strains, etc.

In design problems the question is to determine a system which will show a behaviour accepetable in concordance with some predetermined standards, under some predetermined generalized loading conditions.

Engineers solve these problems with tools falling into two broad classes: Root finding and optimization. The first tool, root finding, is for solving the problems of type:

Determine $\mathbf{x}$, which satisfies

$\mathbf{Ax - b = 0}$ (for linear problems), or

$\mathbf{A(x, b) = 0}$ (for non linear problems).

The second tool is the one for solving optimization problems:

Determine $\mathbf{x}$, which optimizes, i.e. minimizes or maximizes, depending on the formulation,

$F\mathbf{(x, b)}$, where F is a scalar function.

In fact, optimization problems usually come with some constraints. But their existence or inexistence does not change the general characteristics of the problem, although they change the difficulty level considerably.

In structural problems, **x** and **b** in the above formulations reflect generalized displacements and generalized loads, respectively.

These two types of problems and the two types of tools used to solve them can be shown in a two by two matrix as in Figure 1 with approximate weights of usage. It is shown in this figure that, structural design is usually performed by using both root finding techniques and optimization techniques. On the other hand, structural analysis is performed usually by root finding, and exceptionally by optimization. In fact, the aim of this paper to put forward the idea that analysis by optimization should not be represented in such a table by a small x, but by a much larger x.

		Task 1 **ANALYSIS**	**Task 2** **DESIGN**
Tool 1 **Root Finding**	**A(x, b) = 0** **x = ?**	**X**	**X**
Tool 2 **Optimization**	opt { F(**x, b**) } **x = ?**	x	**X**

FIGURE 1 - ALLOCATION OF MATHEMATICAL TOOLS TO SOLVING ENGINEERING PROBLEMS.

STRUCTURAL DESIGN

Structural design is one of the most important tasks of a structural engineer, as it already involves structural analysis and evaluation. Design of a structure starts by choosing shape, dimensions and sizes, followed by the analysis of the resulting structure under the given loads. The analysis, which is usually performed by root finding, gives the stresses and strains in the body, the generalized displacements at the representative points of the structure, and the support reactions. If all these values are within acceptable limits, the structure is economically satisfactory, and constractable, then the designer might stop the calculations and end the designing process. In most of the cases, some modifications in the structura reveal themselves necessary, so that some dimensions and sizes are changed and the analysis is repeated again using the technique of root finding. Theoretically this procedure is repeated as many times as necessary until the designer decides that new impovements are not possible to obtain or that they are negligible. A flow chart for this procedure is given in Figure 2.

The paragraph above shows that root finding is an important tool in design. That is why the cell Design – Root finding has an X in it with an important emphasize.

On the other hand, design can be done by optimization techniques only, without using the technique of root finding. But this necessitates a formulation based on optimization techniques and is applicable only to simple structures. In the literature, there are examples for these kind of solutions. The size of X in the cell Design – Optimization is aimed to reflect these solutions together with the parts of Figure 2 excluding structural analysis.

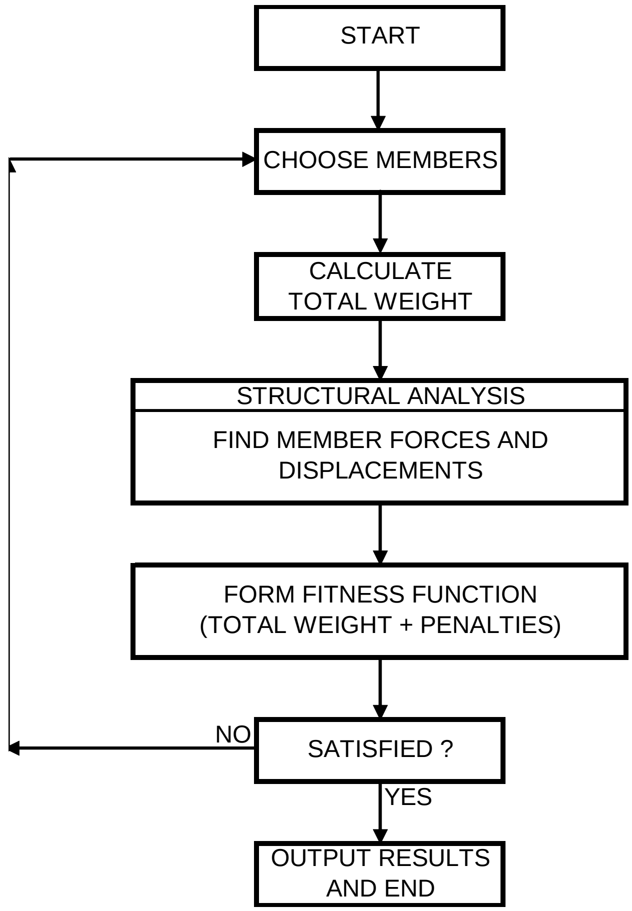

FIGURE 2 - FLOW CHART FOR DESIGN OF A TRUSS

STRUCTURAL ANALYSIS

Analysis of a linearly behaving structure is usually performed by forming the load vector **b,** and the stiffness matrix **A**, which can be done in different ways, thus forming the equation **Ax = b**, or **Ax – b = 0**, and then finding the root **x** of this matrix equation where **x** shows the displacements of the nodal points of truss. In nonlinear problems, the equation to be solved takes the form **A(x, b) = 0**. In this case, the main character of the problem does not change, but it becomes incomparably difficult.

The most advanced technique of structural analysis, the Finite Element Method (FEM), uses this methodology; it presents a uniform and general way of forming the equation **Ax – b = 0** for linear problems. The method is enriched in itself by techniques for solving this matrix equation to find the reactions, displacements and internal effects. For nonlinear problems, the problem is so formulated that the nonlinear equation **A(x, b) = 0** is replaced by linear equations **Ax – b = 0**, valid for successive intervals, and the solution is obtained for example by generalized Newton Raphson Method, which is a root finding technique.

The technique described is the most common method for analysis of structures. That is why the cell Analysis – Root Finding in Figure 1 has a big X in it. The small x in the cell Analysis – Optimization is aimed to represent solutions given in the literature just for the sake of explaining energy methods.

ANALYSIS BY OPTIMIZATION

In fact, the smallness of this x in the cell Analysis – Optimization is practically the subject of the current study. Recent studies have shown that structures can be analysed by using optimization techniques efficiently and without having the shortcomings due to nonlinearity that the formulations leading to the use of root finding technique is accoupled with.

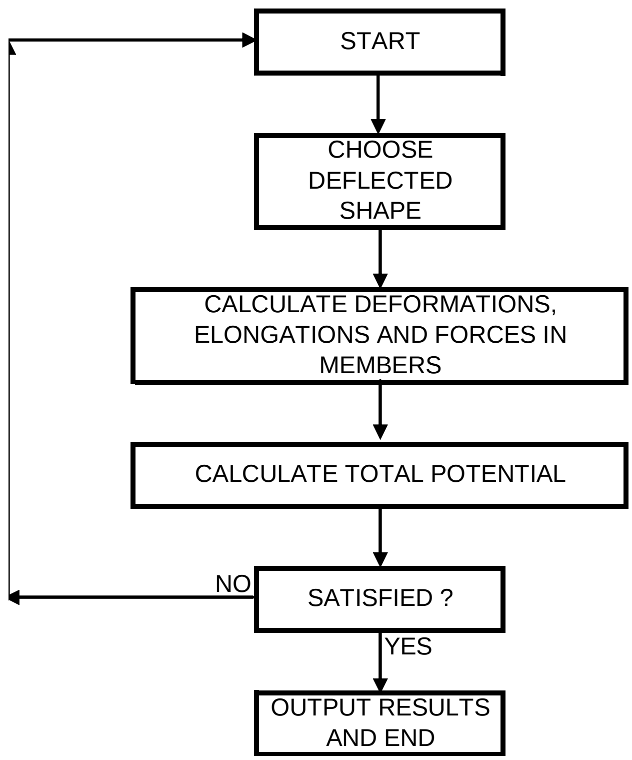

FIGURE 3 - FLOW CHART FOR ANALYSIS OF A TRUSS BY OPTIMIZATION

The theory behind this method is in fact very simple and sound: The configuration corresponding to the equilibrium of a structure is the one with the minimum potential energy. Thus the solution is based on this basic principle of minimum potential energy [Oden 1967, for instance.): Find the configuration that minimizes the total potential energy of a structure. This principle is already used in analysing structures, but in some demonstrative examples, for structures made of a few members and springs. Very recently, it has been shown that the method can be applied to more complicated structures, making use of increased capacities of computers and with the aid of metaheuristic algorithms that opened new horizons in optimization theory [Toklu (2004 a, b, c, d)].

In order to understand the method, consider the problem of analyzing a structure. The total potential U for a given state of deformations (characterized by the strains ε within the body, creating the generalized deflections u_i coupled with the generalized loads P_i) can be written as

$$U(\varepsilon) = \int_V e(\varepsilon)\, dV - \sum_{i=1}^{N_P} P_i u_i$$

$$e(\varepsilon) = \int_0^{\varepsilon} \sigma(\varepsilon)\, d\varepsilon$$

where
σ and ε are stress and strain which are interrelated through σ = σ (ε),
e is the strain energy density,
N_P is the number of loads,
V is the volume of the body.

The integral above gives the strain energy stored in the body, and the summation hitherto is the work done by applied forces and moments, along corresponding general displacements. For a given material the stress strain relation σ = σ (ε) is assumed to be completely known and thus it will be possible to determine the integrals in the above equations.

To simplify the problem, consider a space truss with N_m prismatic members, N_j joints and N_P loads. Consider the element ij with original end coordinates (x_i, y_i, z_i) and (x_j, y_j, z_j) and with original length

$$L(\mathbf{0}) = [(x_j-x_i)^2 + (y_j-y_i)^2 + (z_j-z_i)^2]^{1/2}.$$

After end displacements (u_i, v_i, w_i) and (u_j, v_j, w_j) corresponding to a configuration **c**, the final length will be

$$L(\mathbf{c}) = [(x_j-x_i + u_j-u_i)^2 + (y_j-y_i + v_j-v_i)^2 + (z_j-z_i + w_j-w_i)^2]^{1/2}$$

where $\mathbf{c} = [c_1\ c_2\ \ldots\ c_{3Nm}]^T = [u_1\ v_1\ w_1\ u_2\ v_2\ w_2\ \ldots\ u_{Nm}\ v_{Nm}\ w_{Nm}]^T$ represent the displaced configuration of the structure. The elongation of the member and the uniform strain in it is then:

$$\Delta L(\mathbf{c}) = L(\mathbf{c}) - L(\mathbf{0})$$

$$\varepsilon(\mathbf{c}) = \Delta L(\mathbf{c}) / L(\mathbf{0}).$$

It can be seen that if the end displacements are known, then strain for each member can thereof be determined. Then the integral can be taken for each member to yield e_j, j =1, …,N_m. Since the volume of an original truss element is A_jL_j, U takes the form

$$U(\varepsilon) = \sum_{j=1}^{N_m} e_j A_j L_j - \sum_{i=1}^{N_P} P_i u_i$$

The problem then is to determine the vector $\mathbf{c} = [\, u_1, v_1, \ldots, u_{Nj}, v_{Nj}]^T$ satisfying boundary conditions and minimizing U. The boundary conditions may be considered as the constraints on displacements at supports.

An algorithm is prepared based on this formulation (Figure 3), and a computer program is developed based on local search method [Toklu 2004a]. Various problems are solved using this program to show its effectiveness [Toklu 2004a, b, c, d]. The effectiveness and the accuracy of the method has been shown by following studies [Kaveh, A.; Rahami, H. 2006].

CONCLUSIONS

The method of analyzing structures through optimizationespecially by metaheuristic optimization methods, which rests on very simple but sound principles, is seen to be very powerful in solving any problem on truss statics. Some points to be marked can be listed as follows:

- Geometric nonlinearity and material nonlinearity can be handled very easily.
- There is no difference in formulating statically determinate and indeterminate structures, and even those having geometric instability.
- Accuracy can be controlled to the desired degree, and is never lost.
- The main part of the software is a mere 120 kBytes program. Whatever the size of the problem it is designed to deal with, there will not be a notable increase in this number. Thus the program is very efficient capacity wise and does not require huge memories.
- The algorithm does not involve solving any matrix equations.
- Materials with rupture limits, yield properties, and even those with stress-strain relations changing as a function of the slenderness and the sign of the strain can be tackled with no difficulty.
- Post buckling shapes and snap-through mechanisms of trusses can be determined with no extra effort.

It is obvious that the method can be generalized to more general structural problems. Beams, frames, plates, shells, volumes etc. can be treated in the same way. The method can be generalized even to any problem out of structural engineering field where FEMs are applicable, including fluid mechanics, soil mechanics, electromagnetic field theory, etc. where the equilibrium equations are coupled with a minimum total potential.

Basic characteristic of these applications will be smaller computer memories, easier and less sophisticated programs and larger run times but smaller conception and execution times as compared to similar problems solved by FEM. It is the belief of the author that the robustness and versatility of the method will increase with the time as other types of problems are investigated with different optimization techniques and parameters.

The method, even in this first application, enabled solving problems like trusses which are unstable by definition, which were never solved before without being treated specifically. With its generalizations, this "Total Potential Optimization Method" (TPO) seems to have an immense future.

With the introduction of this method, it seems that the subject of solving structural analysis problems by the mathematical tool of optimization, will be more and more important in practical engineering applications. This will be accoupled with the engineers' being literate of metaheuristic optimization techniques like genetic algorithms and harmony search.

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